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SOME STRONG LAWS FOR COUNTABLE NON-HOMOGENEOUS MARKOV CHAINS

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Abstract

The purpose of this paper is to present some strong laws for the frequency of occurrence of states and ordered couples of states of countable non-homogeneous Markov chains, and put forward a new approach for the study of convergence a.e..

1. **Introduction**. The strong laws for countable non-homogeneous Markov chains were discussed in references [1]-[9], where various restrictions were imposed on the Markov chains. The purpose of this paper is to put forward a new aproach of using the Lebesgue theorem on differentiability of monotone functions to study the convergence "almost every where", and to present some new strong laws for the frequency of occurrence of states and ordered couples of states of the countable non-homogeneous Markov chains.

Throughout this paper we shall deal with the underlying probability space $((0,1),\beta,\mu)$, where β is the class of Borel sets in interval (0,1), and μ is the Lebesgue measure. We first give, in the above probability space, a realization of the Markov chain with state space $S=\{1,2,3,...\}$, initial distribution

$$q_1,\ q_2,\ q_3,\ \ldots, \tag{1}$$

and one-step-transition matrix

$$_nP = \begin{bmatrix} _np_{11} & _np_{12} & _np_{13} & \ldots \\ _np_{21} & _np_{22} & _np_{23} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \quad n=1,2,\ldots \tag{2}$$

Let

$$q_{n_1},\ q_{n_2},\ q_{n_3},\ \ldots \qquad (3)$$

be the positive terms of (1), where $n_1 < n_2 < n_3 < \ldots$. Divide [0,1) into countable many right-semiopen intervals d_{x_0} ($x_0 = n_1, n_2, n_3, \ldots$), that is,

$$d_{n_1} = [0,\ q_{n_1}),\ d_{n_2} = [q_{n_1},\ q_{n_1} + q_{n_2}), \ldots .$$

These intervals will be called the 0-th order d-intervals. Proceeding inductively, suppose the n-th order d-intervals $d_{x_0 \ldots x_n}$ have been defined. Then divide the right-semiopen interval $d_{x_0 \ldots x_n}$ into countable many right-semiopen intervals

$$d_{x_0 \ldots x_n x_{n+1}} \ (x_{n+1} = m_i, \quad i=1,2,3,\ldots)$$

according to the ratio ${}_n p_{x_n m_2} : {}_n p_{x\ m_2};\ {}_n p_{x_n m_3} : \ldots,$ where

$${}_n p_{x_n m_i},\ i=1,2,3,\ldots,\ m_1 < m_2 < m_3 < \ldots \qquad (4)$$

are the positive elements of the x_n-th row of the transition matrix ${}_n P$. In this way the (n+1)st order d-intervals are created. It is easy to see that

$$\mu(d_{x_0 \ldots x_n}) = q_{x_0} \prod_{m=0}^{n-1} {}_m p_{x_m x_{m+1}} \qquad (5)$$

Define, for $n \geq 0$, a random variable X_n: [0,1) -->S as follows:

$$X_n(w) = x_n, \quad \text{if } w \in d_{x_0 \ldots x_n} \qquad (6)$$

By (5) and (6) we have

$$\mu(X_0=x_0, \ldots, X_n=x_n) = \mu(d_{x_0 \ldots x_n}) = q_{x_0} \prod_{m=0}^{n-1} {}_m p_{x_m x_{m+1}} \qquad (7)$$

Therefore $\{X_n,\ n>0\}$ is a Markov chain with initial distribution (1) and transition matrix (2).

In order to prove the theorems below we first construct an auxillary function. Assume

$${}_n R = \begin{bmatrix} {}_n r_{11} & {}_n r_{12} & {}_n r_{13} & \\ {}_n r_{21} & {}_n r_{22} & {}_n r_{23} & \\ {}_n r_{31} & {}_n r_{32} & {}_n r_{33} & \ldots, \\ \ldots & \ldots & \ldots & \end{bmatrix} \quad n=0,1,2,\ldots \qquad (8)$$

is another sequence of one-step transition matrices where ${}_nr_{ij} > 0$ if and only if ${}_np_{ij}>0$. The sequence (8) will be denoted by R. Using the initial distribution (1) in the same procedure as above, except with n^R in place of nP we construct a new collection of intervals $\Delta_{x_0 \ldots x_n}$. It is clear that

$$x_0 = d_{x_0} \; (x_0 = n_1, n_1, n_3, \ldots), \tag{9}$$

and

$$\mu(\Delta_{x_0 \ldots x_n}) = q_{x_0} \prod_{m=0}^{n-1} {}_n r_{x_m x_{m+1}}. \tag{10}$$

Let $d^-_{x_0 x_1 \ldots x_n}$ and $d^+_{x_0 \ldots x_n}$ be, respectively, the left and right end-points of $d_{x_0 \ldots x_n}$; define $\Delta^-_{x_0 \ldots x_n}$ and $\Delta^-_{x_0 \ldots x_n}$ similarly. Let Q be the set of end-points of all d-intervals. Now we define a function f:[0,1)-->[0,1) as follows:

$$f(d^-_{x_0 \ldots x_n}) = \Delta^-_{x_0 \ldots x_n}, \quad f(d^+_{x_0 \ldots x_n}) = \Delta^+_{x_0 \ldots x_n}, \tag{11}$$

$$f(w) = \sup\{f(t), \; t \in Q \cap [0, w)\}, \; w \in [0,1)-Q . \tag{12}$$

It is easy to see that f is increasing on [0,1). Let

$$t_n(R,w) = \frac{\mu(\Delta_{x_0 \ldots x_n})}{\mu(d_{x_0 \ldots x_n})}, \quad w \in d_{x_0 \ldots x_n}. \tag{13}$$

By (5) and (10)-(13) we have

$$t_n(R,w) = \frac{f(d^+_{x_0 \ldots x_n})-f(d^-_{x_0 \ldots x_n})}{d_{x_0 \ldots x_n} - d_{x_0 \ldots x_n}} = \pi \frac{{}_m r_{x_m x_{m+1}}}{{}_m p_{x_m x_{m+1}}}, \; w \in d_{x_0 \ldots x_n}$$

Let k, $1 \in S$, $\lambda > 0$ be a constant. For each $m \geq 0$, when $0 < {}_m p_{kl} < 1$, choose the value of ${}_m r_{kl}$ such that

$$\frac{{}_m r_{kl} \, (1 - {}_m p_{kl})}{{}_m p_{kl} \, (1 - {}_m r_{kl})} = \lambda , \tag{15}$$

that is,

$$ {}_m r_{kl} = \frac{{}_m p_{kl}}{1+(-1) \, {}_m p_{kl}} \tag{16}$$

(it is easy to see that $0<{}_m r_{kl}<1$), and let

$$ {}_m r_{kj} = \frac{1-{}_m r_{kl}}{1-{}_m p_{kl}}\, {}_m p_{kj}\ ,\ j\neq l; \qquad (17)$$

when ${}_m p_{kl} = 0$ or 1, let

$$ {}_m r_{kj} = {}_m p_{kj}\ ,\ j \in S; \qquad (18)$$

when $i \neq k$, let

$$ {}_m r_{ij} = {}_m p_{ij},\ j \in S. \qquad (19)$$

When ${}_m r_{ij}$ $(i,j\in S,\ m=0,1,2,\dots)$ are defined by (15)-(19), the sequence (8) and the function defined by (11) and (12) are determined by λ , so that $t_n(R,w)$ in (13) may be denoted by $t_n(\lambda,w)$.

Assume that $\{X_n, n>0\}$ is defined by (6), $k,l,\in S$, $A_n(k,l,w)$ is the number of occurrence of couple (k,l) in the partial sequence of order couples

$$(X_0(w),\ X_1(w)),\ (X_1,(w),X_2(w)),\dots,\ (X_{n-1}(w),\ X_n(w)), \qquad (20)$$

and $I_k(s)$ $(k=1,2,\dots)$ is a function defined on S:

$$ I_k(s) = \begin{cases} 1, & \text{if } s=k; \\ 0, & \text{if } s\neq k. \end{cases} \qquad (21)$$

Lemma 1. if $w\in d_{x_0\dots x_n}$ $(n\geq 1)$,

$$ t_n(\ ,w) = \lambda^{A_n(k,l,w)} \prod_{m=0}^{n-1} \left(\frac{1}{1+(\ -1)\,{}_m p_{kl}}\right)^{I_k(x_m)} \qquad (22)$$

Proof. There are the three cases for the factors in (14):

Case 1. $x_m \neq k$: by (19) we have

$$ \frac{{}_m r_{x_m x_{m+1}}}{{}_m p_{x_m x_{m+1}}} = 1 \qquad (23)$$

Case 2. $x_m=k$, $x_{m+1}=1$: by(16) and (18) we have

$$\frac{{}_m r_{x_m x_{m+1}}}{{}_m p_{x_m x_{m+1}}} = \frac{\lambda}{1+(\lambda-1)\,{}_m p_{k1}} \qquad (24)$$

Case 3. $x_m=k$, $x_{m+1}\neq 1$: by (17) and the equality

$$\frac{1-{}_m r_{k1}}{1-{}_m p_{k1}} = \frac{1}{1+(\lambda-1)\,{}_m p_{k1}} \quad ({}_m p_{k1}<1)\ ; \qquad (25)$$

(Note. Since $x_m=k$ and ${}_m p_{k1}=1$ implies $x_{m+1}=1$, $x_m=k$ and $x_{m+1}=1$ implies ${}_m p_{k1},1)$ we have

$$\frac{{}_m r_{x_m x_{m+1}}}{{}_m p_{x_m x_{m+1}}} = \frac{1}{1+(\lambda-1)\ {}_m p_{k1}} \qquad (26)$$

By (23), (24), (26) and (24), (22) follows, Q.E.D

For each $w\in[0,1)$, let $x_n=X_n(w)$, then $w\in d_{x_0\ldots x_n}$ $(n\geq 0)$ and we have the rewrite of (22):

$$t_n(\ ,w) = \lambda^{A_n(k,1,w)} \prod_{m=0}^{n-1} \left(\frac{1}{1+(\lambda-1)\,{}_m p_{k1}}\right)^{I_k(X_m(w))}$$

$$w\in[0,1). \qquad (27)$$

2. The strong law for the frequency of occurrence of ordered couples of states.

Theorem 1. let $\{X_n,\ n\geq 0\}$ be a Markov chain defined by (6) $k,1\in S$, $I_k(s)$ and $A_n(k,1,w)$ be defined as before, $\sigma_n(k,1,w)$ $(n=0,1,2,\ldots)$ be positive random variables on $([0,1),\ \beta,\mu)$, and $D(k,1)$ be the set of $w\in[0,1]$ satisfying the following:

$$\text{(i)}\quad \lim_{n\to\infty} \sigma_n\,(k,1,w) = \infty \qquad (28)$$

$$\text{(ii)}\quad \limsup_{n\to\infty} \frac{\sum_{m=0}^{n=1} I_k\ (X_m(w)\ {}_m p_{k1}}{\sigma_n\ (k,1,w)} = \phi\ (w)<\infty, \qquad (29)$$

then

$$\lim_{n\to\infty} \frac{A_n(k,1,w) - \sum_{m=0}^{n-1} I_k\,(X_m(w)\,)\ m p k 1}{\sigma_n(k,1,w)} = 0 \text{ a.e. } x\in D(k,1)$$

(30)

Proof. Let $\lambda > 0$ be constant, R be defined by (15)-(19), function f be defined by (11) and (12), $t_n(\ ,w) = t_n\,(R,w)$ be defined by (13). Letting H(,K,1) be the set of points of differentiability of f, $\mu(H(\ ,k,1)) = 1$ by the Lebesgue theorem on differentiability of monotone functions. Let $w\in H(\ ,k,1)$, and $w\in dx_0\ldots x_n$. If $\lim_{n\to\infty} \mu(dx_0\ldots x_n) = d > 0$,

$$\lim_{n\to\infty} t_n(\lambda,w) = \lim_{n\to\infty} \frac{\mu(\triangle x_0\ldots x_n)}{\mu(dx_0\ldots x_n)} = \frac{\lim_{n\to\infty} \mu(\triangle x_0\ldots x_n)}{d} < \infty,$$

(31)

if $\lim_{n\to\infty} \mu(dx_0\ldots x_n) = 0$, by a property of derivative

(cf. [10], 345-346) we have

$$\lim_{n\to\infty} t_n(\lambda,\ w) = f'(w) < \infty \tag{32}$$

By (31) and (32),

$$\lim_{n\to\infty} t_n\,(\lambda,w) = \text{finite number},\ w\in H\,(\lambda,k,1). \tag{33}$$

Letting $A(\lambda,k,1)=H(\lambda,k,1)\cap D(K,1)$, by (33) and (28),

$$\limsup_{n\to\infty} \frac{1}{\sigma_n(k,1,w)} \ln t_n(\lambda,w)\le 0,\ w\in A(\ ,k,1). \tag{34}$$

By (34) and (27),

$$\limsup_{n\to\infty} \frac{A_n(k,1,w)}{n(k,1,w)} \ln \quad - \frac{1}{\sigma_n(k,1,w)} \sum_{m=0}^{n-1} I_k(X_m(w)).$$

$$\ln(1+(\lambda-1)\ {}_mp_{kl})] \le 0, \quad w\in A(\ ,j,l). \qquad (35)$$

Letting $\lambda > 1$, and dividing the two sides of (35) by $\ln\lambda$, we have

$$\limsup_{n\to\infty} \frac{A_n(k,l,w)}{\sigma_n(k,l,w)} - \frac{1}{\sigma_n(k,l,w)}$$

$$\sum_{m=0}^{n-1} I_k(X_m(w)) \frac{\ln(1+(\lambda-1)_m p_{kl})}{\ln\lambda} \le 0, \quad w\in A(\ ,k,l). \qquad (36)$$

By (36), (29) and the inequality $0 < \ln(1+x) < X$ $(x>0)$, we have

$$\limsup_{n\to\infty} \frac{A_n(k,l,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{kl}}{\sigma_n(k,l,w)} \le$$

$$\limsup_{n\to\infty} \frac{1}{\sigma_n(k,l,w)} \sum_{m=0}^{n-1} I_k(X_m(w)) \frac{\ln(1+(\lambda-1)_m p_{kl}}{\ln\lambda} - {}_mp_{kl}$$

$$\limsup_{n\to\infty} \frac{1}{\sigma_n(k,l,w)} \sum_{m=0}^{n-1} I_k(X_m(w)) \left(\frac{\lambda-1}{\ln\lambda} - 1\right) {}_mp_{kl}$$

$$\le \left(\frac{\lambda-1}{\ln\lambda} - 1\right) \phi(w), \quad w\in A(\ ,k,l). \qquad (37)$$

Assuming $\lambda_i>1$ $(i=1,2,\ldots,)$, $i\to 1+0$ $(i\to\infty)$, and letting

$$A(k,l) = \bigcap_{i=1}^{\infty} A(\lambda_i,k,l)$$

then for each $i=1,2,\ldots,$ we have by (37),

$$\limsup \frac{A_n(k,l,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{kl}}{\sigma_n(k,l,w)} \le \frac{\lambda_i - 1}{\ln\lambda_i} - 1)\phi(w),$$

$$w \in A(k,l). \qquad (38)$$

Since $\frac{\lambda_i - 1}{\ln\lambda_i} \to 1$ $(i \to \infty)$, we have by (38),

$$\limsup \frac{A_n(k,1,w) - \sum_{m=0}^{n-1} I_k(S_m(w))\ {}_mp_{k1}}{\sigma_n(k,1,w)} \le 0, \quad w \in A(k,1) \qquad (39)$$

Similarly, letting $0 < \lambda < 1$, and dividing the two sides of (35) by $\ln\lambda$, we have

$$\liminf_{n\to\infty} \frac{A_n(k,1,w)}{\sigma_n(k,1,w)} - \frac{1}{\sigma_n(k,1,w)} \sum_{m=0}^{n-1} I_k(X_m(w)) \frac{\ln(1+\lambda-1)\ {}_mp_{k1}}{\ln\lambda}$$

$$\ge 0, \quad w \in A(\ ,k,1). \qquad (40)$$

By (40), (29) and the inequality $\ln(1+x) \le$ $(-1<x<0)$, we have

$$\liminf_{n\to\infty} \frac{A_n(k,1,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{k1}}{\sigma_n(k,1,w)} \ge \liminf \frac{1}{\sigma_n(k,1,w)}$$

$$\sum_{m=0}^{n-1} I_k(X_m(w)) \left(\frac{\ln(1+(\lambda-1)\ {}_mp_{k1}}{\ln\lambda} - {}_mp_{k1} \right)$$

$$\ge \liminf_{n\to\infty} \frac{1}{\sigma_n(k,1,w)} \sum_{m=0}^{n-1} I_k(X_m(w)) \left(\frac{\lambda-1}{\ln\lambda} - 1 \right) {}_mp_{k1}$$

$$\ge \left(\frac{\lambda-1}{\ln\lambda} - 1 \right) \phi(w), \qquad w \in A(\lambda,k,1). \qquad (41)$$

Assuming $0 < \tau_i < 1$ $(i=1,2,\ldots)$, $\tau_i \to 1-0$ $(i \to \infty)$, and letting

$$B(k,1) = \bigcap_{i=1}^{\infty} A(\tau_i,k,1) ,$$

then for each i=1,2,..., we have by (41),

$$\liminf_{n\to\infty} \frac{A_n(k,l,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{kl}}{\sigma_n(k,l,w)} \geq \left(\frac{\tau_i - 1}{\ln \tau_i} - 1\right)\phi(w),$$

$w \in B(k,l)$. (42)

Since $\frac{\tau_i - 1}{\ln \tau_i} \to 1$ $(i \to \infty)$, we have by (42),

$$\liminf \frac{A_n(k,l,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{kl}}{\sigma_n(k,l,w)} \geq 0, \quad w \in B(k,l) \qquad (43)$$

Letting $C(k,l)=A(k,l) \cap B(k,l)$, we have by (39) and (43),

$$\lim_{n\to\infty} \frac{A_n(k,l,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{kl}}{\sigma_n(k,l,w)} = 0, \quad w \in C(k,l). \qquad (44)$$

Since $C(k,l)$ $D(k,l)$, and $\mu(C(k,l)) = \mu(D(k,l))$, hence, by (44), (30) is true. Q.E.D.

Choosing $\sigma_n(k,l,w)$ appro-priately, we have two immediate consequences below, where no restriction is imposed on the Markov chains.

Theorem 2. Let $\{X_n,\ n \geq 0\}$ be a Markov chain defined by (6), $k,l,\in S$, $I_k(s)$ and $A_n(k,l,w)$ be defined as before, and let

$$S_n(k,w) = \sum_{m=0}^{n-1} I_k(X_m(w)), \qquad (45)$$

that is, $S_n(k,w)$ be the number of occurence of k in the sequence $X_0(w), X_1(w), \ldots, X_{n-1}(w)$. Then

$$\lim \frac{A_n(k,l,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{kl}}{S_n(k,w)} = 0 \quad \text{a.e., } w \in D_k \qquad (46)$$

where

$$D_n = \{w: \lim_{n\to\infty} S_n(k,w) = \infty\}. \qquad (47)$$

Proof: Letting $\sigma_n(k,1,w) = S_n(k,w)$ in Theorem 1, since

$$\frac{\sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{k1}}{S_n(k.w)} \le 1,\ n=0,1,2,\ldots,$$

hence $D=(k,1)=D_k$, and (46) follows by (30). Q.E.D

Corollary. If

$$\lim_{n\to\infty} {}_n p_{k1} = p_{k1} \qquad (48)$$

then

$$\lim_{n\to\infty} \frac{A_n(k,1,w)}{S_n(k,w)} = p_{k1} \qquad \text{a.e.,}\ w\in D_k. \qquad (49)$$

Proof. Since (48) implies

$$\lim_{n\to\infty} \frac{\sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{k1}}{S_n(k,w)} = p_{k1},\ w\in D_k$$

(49) follows by (46).

Theorem 3. Let $\{X_n,\ n\ge 0\}$ be a Markov chain defined by (6), $k,1\in S$, $I_k(s)$ and $A_n(k,1,w)$ be defined as before, and let

$$H(k,1) = \{w: \lim_{n\to\infty} \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{k1} = \infty\}.$$

Then

$$\lim_{n\to\infty} \frac{A_n(k,1,w) - \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{k1}}{\sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{k1}} = \text{a,e.,} \qquad w\in H(k,1). \qquad (50)$$

Proof. Letting $\sigma_n(k,1,w) = \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_mp_{k1}$ in Theorem 1, follows easily.

3. The strong law for the frequence of occurrence of states.

In order to prove the theorems below we construct another auxiliary function. For this purpose, we first give another construction of matrix R in (8). Let 1 ∈S, >0 be a constant. For each $m \geq 0$ and all $k \in S$, let ${}_mr_{ij}$ be defined by (16)-(18). Hence the sequence (8) and the function f defined by (11) and (12) are determined by λ, and we will denote $t_n(R, w)$ in (13) by $t_n(\lambda, w)$

Lemma 2. If $w \in d_{x_0 \ldots x_n}$ $(n \geq 1)$,

$$t_n(\lambda, w) = \frac{S_n(1,w)+I_1(x_n)-I_1(x_0)}{\prod_k \prod_{m=0}^{n-1} \left(\frac{1}{1+(\lambda-1)\,{}_mp_{k1}} \right)^{I_k(x_m)}}, \tag{51}$$

where the notation $\prod_k$ denotes multiplication over all $k \in S$.

Proof. If $w \in d_{x_0 \ldots x_n}$, we have by (14),

$$t_n(\lambda, w) = \prod_{m=0}^{n-1} \frac{{}_mr_{x_m x_{m+1}}}{{}_mp\quad {}_mp_{x_m x_{m+1}}} = \prod_k \prod_{\substack{m=0 \\ \alpha_m = k}}^{n-1} \frac{{}_mr_{x_m x_{m+1}}}{{}_mp_{x_m x_{m+1}}}. \tag{52}$$

By similar argument, as in the case of Lemma 1, we have

$$\prod_{\substack{x=0 \\ x_m=k}}^{n-1} \frac{{}_mr_{x_m x_{m+1}}}{{}_mp_{x_m x_{m+1}}} = \lambda^{A_n(k,1,w)} \prod_{m=0}^{n-1} \left(\frac{1}{1+(\ -1)\,{}_mp_{k1}} \right)^{I_k(x_m)} \tag{53}$$

By (52), (53) and the equality

$$\sum_k A_n(k,1,w) = S_n(1,w)-I_1(x_n)-I_1(x_0),\ w\in d_{x_0,\ldots x_n} \qquad (54)$$

where the notation $\sum_k$ denotes summation over all $k\in S$, (51) follows.

We have the rewrite of (51):

$$t_n(\lambda,w) = S_n(1,w)+I_1(X_n(w))-I_1(X_0(w))$$

$$\prod_k \prod_{m=0}^{n-1} \left(\frac{1}{1+(\lambda-1)\ {}_m p_{k1}}\right)^{I_k(X_m(w))},\quad w\in[0,1) \qquad (55)$$

as in the case of (27).

Theorem 4. Let $\{X_n,\ n\geq 0\}$ be a Markov chain defined by (6), $I_k(s)$ and $S_n(1,w)$ be defined as before, $\sigma_n(1,w)$ $(n=0,1,2,\ldots)$ be positive random variables on $([0,1),\ \beta,\mu)$ and $D(1)$ be the set of $w\in[0,1)$ satisfying the following:

$$\text{(1)}\quad \lim_{n\to\infty} \sigma_n(1,w) = \infty; \qquad (56)$$

$$\text{(11)}\quad \limsup_{n\to\infty} \frac{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_m p_{k1}}{\sigma_n(1,w)} = \phi(w) < \infty, \qquad (57)$$

then

$$\lim_{n\to\infty} \frac{S_n(1,w)-\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_m p_{k1}}{\sigma_n(1,w)} = 0 \text{ a.e., } w\in D(1) \qquad (58)$$

Proof. Letting $H(\lambda,1)$ be the et of points of differentiability of function f discussed in lemma 2, $\mu H(\lambda,1)=1$. We have, as in the case of (33),

$$\lim_{n\to\infty} t_n(\lambda,w)=\text{finite number},\quad w\in H(\lambda,1). \qquad (59)$$

Let $A(\lambda,1)=H(\lambda,1)\cap D(1)$. Then

$$\limsup_{n\to\infty} \frac{1}{\sigma_n(1,w)} \ln t_n(\lambda,w) \le 0, \quad w \in A(\lambda,1)$$

by (59) and (56), and (60)

$$\limsup_{n\to\infty} \left[\frac{S_n(1,w)}{\sigma_n(1,w)} \ln\lambda - \frac{1}{\sigma_n(1,w)} \sum_k \sum_{m=0}^{n-1} I_k(X_m(w) \ln(1+(\lambda-1)\ {}_mp_{k1}) \right]$$

$$\le 0, \quad A \in A(\lambda,1) \qquad (61)$$

Letting $\lambda > 1$, and dividing the two sides of (61) by ln , we have

$$\limsup_{n\to\infty} \left[\frac{S_n(1,w)}{\sigma_n(1,w)} - \frac{1}{\sigma_n(1,w)} \sum_k \sum_{m=o}^{n=1} \frac{I_k(X_m(w)) \ln(1+(\lambda-1)\ {}_mp_{k1})}{\ln\lambda} \right]$$

$$\le 0, \quad w \in A(\ ,1) \qquad (62)$$

By (62), (57) and the inequality $0 \le \ln(1+x) \le x (x \ge 0)$, we have

$$\limsup_{n\to\infty} \frac{S_n(1,w) - \sum_k \sum_{m=0}^{n-1} I_k(X_m(w)\ )\ {}_mp_{k1}}{\sigma_n(1,w)}$$

$$\le \limsup_{n\to\infty} \frac{1}{\sigma_n(1,w)} \sum_k \sum_{m=0}^{n-1} I_k(X_m(w)) \left(\frac{\ln(1+(\lambda-1)\ {}_mp_{k1}}{\ln\lambda} - {}_mp_{k1} \right)$$

$$\le \limsup_{n\to\infty} \frac{1}{\sigma_n(1,w)} \sum_k \sum_{m=0}^{n-1} I^k(X^m(w) \ \frac{(\lambda-1)}{\ln\lambda} - 1 \quad {}_mp_{k1}$$

$$\le \frac{\lambda-1}{\ln\lambda} - 1) \ \phi(w), \qquad w \in A(\lambda,1) . \qquad (63)$$

Assuming $\lambda_i > 1$ $(i=1,2,\ldots,)$, $i \to 1+0$ $(i\to\infty)$, and letting

$$A(1) = \bigcap_{i=1}^{\infty} A(\lambda_i,1),$$

then for each $i = 1,2,\ldots,$ we have by (63),

$$\limsup_{n\to} \frac{S_n(1,w) - \sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\, m p_{k1}}{\sigma_n(1,w)} \le \frac{\lambda_i - 1}{\ln \lambda_i}\, \phi(w), \quad w \in A(1) \qquad (64)$$

Since $\dfrac{\lambda_i - 1}{\ln \lambda_i} \longrightarrow 1 \ (1\to\infty)$, we have by (64),

$$\limsup_{n\to\infty} \frac{S_n(1,w) \quad \sum_k \sum_{k=1}^{n-1} I_k(X_m(w)\ )\, m p_{k1}}{\sigma_n(1,w)} \le 0 , \quad w \in A(1) \qquad (65)$$

Similarly, letting $0 < \lambda < 1$, and dividing the two sides or (61) by $\ln$, we have

$$\liminf_{n\to\infty} \frac{S_n(1,w)}{\sigma_n(1,w)} - \frac{1}{\sigma_n(1,w)} \sum_k \sum_{m=0}^{n-1} \frac{I_k(X_m(w))\ln(1+(\lambda-1)\, m p_{k1}}{\ln\lambda} \ge 0,$$

$$w \in A(\lambda,1) \qquad (66)$$

Assuming $0 < \tau_i < 1$ $(i=1,2,\ldots)$, $\tau_i \to 1-0$ $(i-\infty)$, and letting

$$B(1) = \bigcap_{i=1}^{\infty} A(\tau_i, 1)$$

We have by (66),

$$\liminf_{n\to\infty} \frac{S_n(1,w) - \sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\, m p_{k1}}{\sigma^n(1,w)} \ge 0, \quad w \in B(1) \qquad (67)$$

as in the case of (65).

Letting $C(1)=A(1)\cap B(1)$, we have by (65) and (67),

$$\lim_{n\to\infty} \frac{S_n(1,w) - \sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_mp_{k1}}{\sigma_n(1,w)} = 0,\ w\in (1). \qquad (68)$$

Since $C(1) \subset D(1)$ and $\mu(C(1) = \mu(D(1)\ )$, (58) follows by (68).

Theorem 5. Let $\{X_n, n \geq 0\}$ be a Markov chain defined by (6), $I_k(s)$ and $S_n(1,w)$ be defined as before. Then

$$\lim_{n\to\infty} \frac{S_n(1,w)-\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_mp_{k1}}{n} = 0 \text{ a.e. in } [0,1). \qquad (69)$$

Proof. Letting $\sigma_n(1,w) = n$ in Theorem 4, and noticing the inequality
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$$\sum_k \sum_{m=0}^{n-1} I_k(X_m(w)\ )\, {}_mp_{k1} \leq \sum_k \sum_{m=0}^{n=1} \sum I_k(X_m(w)\ ) = n$$

(69) follows by Theorem 4.

Letting $\sigma_n(1,w) = \sum \sum_{m=0}^{n-1} I_k\ (X_m(w)\ )\, {}_mp_{k1}$ in Theorem 4, we have the immediate consequance below.

Theorem 6. Let $\{X_n, n \geq 0\}$ be a Markov chain defined by (6), $I_k(s)$ and $S_n(1,w)$ be defined as before, and let

$$H(1) = \{w:\ \lim_{n\to\infty} \sum_k \sum_{k=0}^{n-1} I_k(X_m(w))\, {}_mp_{k1} = \infty\ \} \qquad (70)$$

Then

$$\lim_{n\to\infty} \frac{S_n(1,w)}{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w)\, {}_mp_{k1}} =1 \text{ a.e., } w \in H(1). \qquad (71)$$

Theorem 7. Let $\{X_n, n \geq 0\}$ be a Markov chain defined by (6), $I_k(s)$ and $S_n(1,w)$ be defined as before , and let

$$S(1) = \{w: \lim S_n(1,w) = \infty\}. \quad (72)$$

Then

$$\lim_{n\to\infty} \frac{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{k1}}{S_n(1,w)} = 1 \quad \text{ae., } w \in S(1) \quad (73)$$

Proof Letting $\sigma_n(1,w) = S_n(1,w)$ in Theorem 4, and $H(1)$ be defined by (70). If $w \in ([0,1)-H(1)\cap S(1)$,

$$\lim_{n\to\infty} \frac{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{k1}}{S_n(1,w)} = 0 \quad (74)$$

and

$$\lim_{n\to\infty} \frac{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{k1}}{S_n(1,w)} = 1 \text{ a.e., } w \in S(1) \cap H(1) \quad (75)$$

Then by (74) and (75),

$$\lim_{n\to\infty} \frac{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{k1}}{S_n(1,w)} = 1 \quad \text{a.e., } w \in S(1). \quad (76)$$

Let $W(1)$ be the set $w \in S(1)$ satisfying the inequality in (76).

Then by (72) and (76) and Theorem 4,

$$\lim_{n\to\infty} \frac{\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\ {}_mp_{k1}}{S_n(1,w)} = 1 \text{ a.e. } w \in W(1). \quad (77)$$

Since $W(1) \subset S(1)$ and $\mu(W(1) = \mu(S(1)))$, (73) follows by 77.

By Theorem 6 and Theorem 7, we have the immediate consequence below.

Theorem 8. Let $\{X_n, n \geq 0\}$ be a Markov chain defined by (6), $S_n(1,w)$ be defined as before. Then

$$\lim_{n\to\infty} S_n(1,w) = \infty \quad \text{a,e. } w \in [0,1) \tag{78}$$

if and only if

$$\lim_{n\to\infty} \sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\, {}_m p_{kl} = \infty \text{ a.e. } w \in [0,1). \tag{79}$$

Corollary. Assume that the Markov chain defined by (6) is homogeneous, that is, ${}_m p_{kl} = p_{kl}$ $(m=0,1,2,\ldots)$, Then (78) holds if and only if

$$\lim \sum_k p_{kl} S_n(k,w) = \infty \quad \text{a.e., } w \in [0,1). \tag{80}$$

Proof. By Theorem 8, and noticing that

$$\sum_k \sum_{m=0}^{n-1} I_k(X_m(w))\, p_{kl} = \sum_k p_{kl} S_n(k,w),$$

the corollary follows.

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